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An example of non-uniqueness in the two-dimensional linear water wave problem

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An example of non-uniqueness in the two-dimensional, linear water wave problem is obtained by constructing a potential which does not radiate any waves to infinity and whose streamline pattern represents the flow around two surface-piercing bodies. The potential is constructed from two wave sources which are positioned in the free surface in such a way that the waves radiated from each source cancel at infinity. A numerical calculation of the streamline pattern indicates that there are at least two streamlines which represent surface-piercing bodies, each of which encloses a source point. A proof of the existence of these lines is then given.

1. Introduction

One of the classic problems in the study of the interaction of water waves with obstacles using linear theory is to determine whether the problem posed in the frequency domain has a unique solution at all frequencies. John (1950) established uniqueness for a class of single, surface-piercing bodies which have the property that any vertical line emanating from the free surface does not intersect the body. His result can be immediately generalized to include extra bodies or variations in the bottom topography, provided that they can all be contained below the original body. However, the result does not cover all bodies and, in particular, it fails for submerged bodies.

A general result for submerged bodies has not been proved and indeed some of the earliest work concentrated on specific geometries. Ursell (1950) established the uniqueness of the solution for a circular cylinder which is submerged in fluid of infinite depth, by using the theory of complex variables. Uniqueness has also been established for the axisymmetric problem for a submerged sphere by Livchitz (1974). However, the methods used in these cases are tailored to specific geometries and it is not easy to see how they can be extended to arbitrary body shapes. Kreisel (1949) used complex variable theory to look at two-dimensional fluid layers of variable depth. By mapping an infinite strip onto the fluid domain, he established uniqueness for a class of layers which in some sense are close to the infinite strip. Other work on fluid of variable depth is given in Vainberg & Maz'ja (1973). They established uniqueness for topographies which satisfy certain geometric conditions but their criteria do not cover all fluid layers. Further conditions were given by Fitzgerald & Grimshaw (1979) who extended the results of Kreisel (1949) to two-dimensional layers which have different depths at either infinity.

A breakthrough for submerged, compact bodies was made by Maz'ja (1978). He derived a very general vector identity involving the velocity potential and its derivatives and established geometric restrictions on the bodies under which the identity could be written as a sum of non-negative integrals equalling zero. This led to a proof that, for such bodies, any velocity potential which satisfies homogeneous boundary conditions must be identically equal to zero throughout the fluid. His work was amplified and extended by Hulme (1984) who produced specific examples of bodies which satisfy Maz'ja's criterion for uniqueness. The number of geometries for which uniqueness can be proved using this method has since been extended by Weck (1990) and Kuznetsov (1991). At about the same time that Hulme publicized the work of Maz'ja, Simon & Ursell (1984) generalized the result of John (1950) to prove uniqueness for two-dimensional bodies which are contained between lines which emanate from the free surface at a certain angle. In particular, they proved that in infinite depth, if one or more submerged bodies are contained between two lines which intersect at a point on the free surface and are inclined at angles $\pm\pi/4$ to the horizontal, then the solution is unique. In general, their results were sometimes stronger and sometimes weaker than the ones described by Hulme (1984) but neither method produces a general proof of uniqueness.

Results concerning more general classes of bodies are available but they are frequency dependent. Beale (1977) showed that for a floating body in an ocean of uniform depth, the solution is unique except possibly for a discrete set of frequencies. This work was extended by Vullierme-Ledard (1983) who also showed that for bodies submerged in fluid of infinite depth, neither zero frequency nor infinite frequency are accumulation points. Her results were confirmed by Simon & Ursell (1984) who also generated bounds for the frequency parameter for which non-uniqueness could occur. Further extensions of the work of Beale (1977) were made by Athanassoulis (1987) and Athanassoulis & Politis (1990).

Much less work has been done on establishing uniqueness for multiple surface-piercing bodies in two dimensions, partly because of the difficulty of dealing with the portions of the free surface which are trapped between the bodies. To the author's knowledge, the only explicit solution which exists for two surface-piercing bodies is that derived by Levine & Rodemich (1958) for a pair of vertical barriers. Kuznetsov (1988) and Kuznetsov & Simon (1995) have generated some frequency-dependent results for two surface-piercing cylinders in two dimensions. By transforming to bipolar coordinates, they show that the solution is unique for all frequencies below a value which depends on the geometry of the bodies. However, they are unable to prove uniqueness for all bodies at all frequencies and, in fact, the purpose of this work is to construct an explicit example of two surface-piercing bodies for which the potential is non-unique at a specific frequency.

A statement of the problem is given in §2. The question of uniqueness can be reduced to the problem of showing whether a certain homogeneous boundary value problem has only the trivial solution $\phi \equiv 0$. An example of non-uniqueness is generated in §3 by constructing a potential which does not radiate any waves to infinity and then interpreting two of the streamlines as body contours. This approach has been used before by Bessho (1965) and more recently by Kyozuka & Yoshida (1981) to generate bodies which have zero radiation damping at isolated frequencies. Here, the potential is constructed from two wave sources which are positioned in such a way that the waves emanating from each cancel at infinity. A similar wave-free potential was constructed by Morris (1974) who looked at configurations of sources placed above a sloping beach which do not radiate waves to infinity. The streamline pattern is generated numerically and indicates that at least two of the streamlines represent surface-piercing bodies, each of which contains a source point. A proof that such streamlines exist is then given in §4.

2. Statement of the problem

The uniqueness problem in linear water waves is formulated by considering the irrotational motion of an inviscid and incompressible fluid. The wave and body motions are assumed to be small compared to the wavelength and so the free surface and body boundary conditions are linearized about their mean positions. In two dimensions, a time-harmonic solution for the velocity potential is sought in the form $\mathrm{Re}\{\phi(x,y)\mathrm{e}^{-\mathrm{i}\omega t}\}$, where ω is the angular frequency and x and y are rectangular Cartesian coordinates with the origin in the mean free surface and the y-axis pointing vertically downwards. If $\phi_1(x,y)$ and $\phi_2(x,y)$ are two potentials which satisfy the same forced boundary value problem, then the difference potential $\phi = \phi_1 - \phi_2$ satisfies

$$\nabla^2\phi = 0 \tag{2.1}$$

in the fluid with the free surface boundary condition

$$K\phi + \frac{\partial\phi}{\partial y} = 0 \qquad \text{on} \qquad y = 0, \tag{2.2}$$

where $K = \omega^2/g$ and g is the acceleration due to gravity. On any body surface

$$\frac{\partial\phi}{\partial n} = 0, \tag{2.3}$$

where $\partial/\partial n$ is the derivative in a direction normal to the body. Under the assumption that the fluid has infinite depth

$$\nabla\phi \to 0 \qquad \text{as} \qquad y \to \infty. \tag{2.4}$$

In addition, ϕ satisfies the radiation condition

$$\frac{\partial\phi}{\partial x} \mp \mathrm{i}K\phi \to 0 \qquad \text{as} \qquad x \to \pm\infty. \tag{2.5}$$

An application of Green's theorem to ϕ and its complex conjugate shows that this homogeneous problem for ϕ cannot produce any outgoing waves and so the radiation condition (2.5) and the large-depth condition (2.4) may be replaced by

$$\phi \to 0 \qquad \text{as} \qquad x^2 + y^2 \to \infty, \quad y \geqslant 0. \tag{2.6}$$

More precisely, Ursell (1950) showed that if there are no waves at infinity, the potential may be expanded in terms of wave-free potentials and, in general, the leading-order behaviour of ϕ as $x^2 + y^2 \to \infty$ is given by a multiple of the lowest symmetric wave-free potential χ_1, which is defined by

$$\chi_1 = \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{Ky}{x^2 + y^2}, \qquad x^2 + y^2 \neq 0. \tag{2.7}$$

The problem of establishing uniqueness is equivalent to showing that the above boundary value problem only has the trivial solution $\phi \equiv 0$, where ϕ is assumed to have continuous second derivatives in the fluid and to be continuous onto the boundary. The solution of a forced problem is therefore non-unique at a particular frequency if a non-zero solution of the homogeneous boundary value problem is found. In the next section, such a solution is constructed.

3. Construction of a non-zero solution of the homogeneous boundary value problem

Rather than consider a specific geometry and look for frequencies at which non-trivial solutions occur, an example of non-uniqueness is derived by constructing a potential which decays appropriately at infinity and interpreting some of the streamlines as body contours. This is similar to the approach taken by Bessho (1965) and Kyozuka & Yoshida (1981) who looked for bodies for which zero damping occurs at certain frequencies.

The potential is constructed from a wave source (see Ursell 1949) placed at the point $(a, 0)$ and another source of equal strength placed at the point $(-a, 0)$, where $Ka = \pi/2$. Thus,

$$\phi = \int_0^\infty \frac{\mathrm{e}^{-ky}}{k-K} \cos k(x-a)\,\mathrm{d}k + \int_0^\infty \frac{\mathrm{e}^{-ky}}{k-K} \cos k(x+a)\,\mathrm{d}k, \tag{3.1}$$

where the contour of integration passes below the poles at $k = K$ in both integrals. This particular combination of sources is chosen because the waves radiated from each source cancel each other out as $|x| \to \infty$ and so ϕ satisfies (2.6). The potential represents a solution to the homogeneous problem for two surface-piercing bodies if at least one of the streamlines of the flow connects the free surface on either side of a source point and another streamline similarly surrounds the other source point. If they exist, such contours may be interpreted as the boundaries of bodies and the potential then has no singularities in the fluid domain because the source points are contained within the interior of the bodies. Physically, the potential is suggestive of a trapped oscillation because, although the waves emanating from the sources cancel as $|x| \to \infty$, they reinforce between the sources to produce a localized standing wave. (This argument should only be used as a guide to the behaviour of the flow field, however, because it relies on using the far-field representation of the source potentials in the region between the sources, which is not strictly valid.)

An alternative representation for ϕ in terms of the exponential integral may be derived which is more convenient for both analytic and numerical work. By deforming the contour of integration into the upper or lower half-plane, depending on the sign of X, it may be shown that

$$⨍_0^\infty \frac{\mathrm{e}^{-ky+\mathrm{i}kX}}{k-K}\,\mathrm{d}k = \begin{cases} \mathrm{e}^{-Ky+\mathrm{i}KX}(-\pi\mathrm{i} + \mathrm{E}_1(-Ky+\mathrm{i}KX)), & X<0,\ y \geqslant 0, \\ -\mathrm{e}^{-Ky}\mathrm{Ei}(Ky), & X=0,\ y>0, \\ \mathrm{e}^{-Ky+\mathrm{i}KX}(\pi\mathrm{i} + \mathrm{E}_1(-Ky+\mathrm{i}KX)), & X>0,\ y \geqslant 0, \end{cases} \tag{3.2}$$

where the integral on the left-hand side of (3.2) is interpreted in a principal value sense. The functions E_1 and Ei are the forms of the exponential integral defined by Abramowitz & Stegun (1965, 5.1.1-2) as

$$\mathrm{E}_1(z) = \int_z^\infty \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t, \qquad |\arg z| < \pi \tag{3.3}$$

and

$$\mathrm{Ei}(Y) = -⨍_{-Y}^\infty \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t, \qquad Y > 0. \tag{3.4}$$

(Despite its appearance, the representation on the right-hand side of (3.2) is continuous on the line $X = 0$, $y > 0$ because of the jump in $\mathrm{E}_1(-Ky + \mathrm{i}KX)$ across this line.) Since $Ka = \pi/2$, the contributions to the integrals in (3.1) from the contours below the poles cancel and so the contour integrals may be replaced by principal value

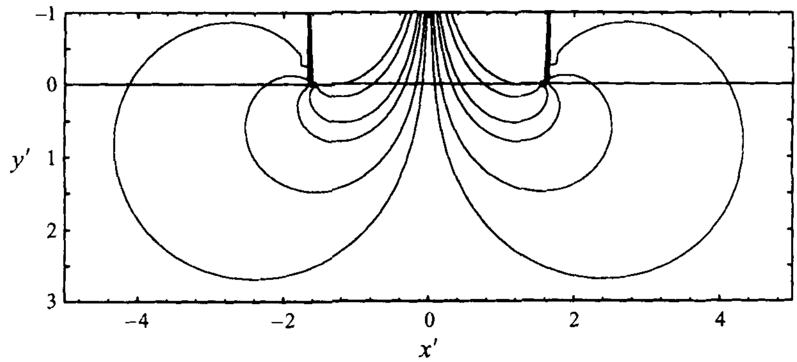

FIGURE 1. The streamline pattern for two wave sources, a non-dimensional distance π apart.

integrals. Once this has been done, it is clear that ϕ is purely real and so the real part of (3.2) may be substituted into (3.1) to give, after some rearrangement,

$$\phi = \mathrm{Re}[\mathrm{e}^{-Ky+\mathrm{i}Kx}(\pi\mathrm{sgn}(Kx-\pi/2) - \pi\mathrm{sgn}(Kx+\pi/2) \\ + \mathrm{i}\mathrm{E}_1(-Ky+\mathrm{i}Kx+\mathrm{i}\pi/2) - \mathrm{i}\mathrm{E}_1(-Ky+\mathrm{i}Kx-\mathrm{i}\pi/2))], \qquad Kx \neq \pm\pi/2 \quad (3.5)$$

and ϕ is continuous on $Kx = \pm\pi/2$, $y > 0$.

The harmonic conjugate to ϕ, hereafter referred to as the stream function ψ, is given by

$$\psi = ⨍_0^\infty \frac{\mathrm{e}^{-ky}}{k-K} \sin k(x-a)\,\mathrm{d}k + ⨍_0^\infty \frac{\mathrm{e}^{-ky}}{k-K} \sin k(x+a)\,\mathrm{d}k. \qquad (3.6)$$

The contribution to the integrals in (3.6) from the contours below the poles cancel and so the stream function is real and may be written in terms of the exponential integral as

$$\psi = \mathrm{Im}[\mathrm{e}^{-Ky+\mathrm{i}Kx}(\pi\mathrm{sgn}(Kx-\pi/2) - \pi\mathrm{sgn}(Kx+\pi/2) \\ + \mathrm{i}\mathrm{E}_1(-Ky+\mathrm{i}Kx+\mathrm{i}\pi/2) - \mathrm{i}\mathrm{E}_1(-Ky+\mathrm{i}Kx-\mathrm{i}\pi/2))], \qquad Kx \neq \pm\pi/2 \quad (3.7)$$

and ψ is continuous on $Kx = \pm\pi/2$, $y > 0$. The streamlines of the flow (lines on which ψ = const.) represent impenetrable barriers in the fluid and may be interpreted as the boundaries of bodies. A numerical calculation of the streamline pattern, using *Mathematica*, is given in figure 1. The x- and y-coordinates have been non-dimensionalized by introducing the new variables $x' = Kx$ and $y' = Ky$ and so the source points are at $(\pm\pi/2, 0)$. In order to illustrate exactly how the streamlines intersect the free surface, the continuation of the streamline pattern into the region $y' < 0$ is given. No physical meaning is attributed to the streamlines in this region and, indeed, the stream function has branch cuts on the lines $x' = \pm\pi/2, y' < 0$. However, from figure 1, it is clear that there are some streamlines which connect the free surface on either side of each source point and which may be interpreted as surface-piercing bodies. Not all streamlines may be used, as some go into a source point from below the free surface, but an illustration is given in figure 2 of two surface-piercing bodies which are generated from parts of the streamlines on which $\psi = \pm 1$. As the spatial variables have been non-dimensionalized by using the wavenumber K, the actual position of the bodies changes as K varies. However, this is to be expected because for a given geometry non-uniqueness can only occur at isolated frequencies.

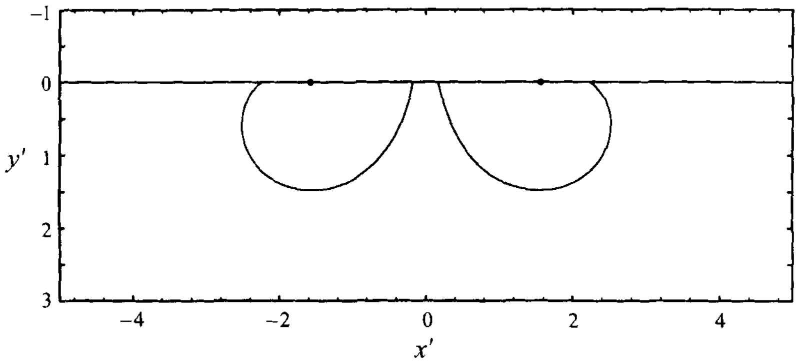

FIGURE 2. Two surface-piercing bodies for which non-uniqueness occurs.

Once two streamlines have been interpreted as bodies, the remaining streamlines in the fluid domain represent lines tangent to the fluid velocity. Thus, from figure 1, the potential may be interpreted as representing a simple heaving of the fluid in between the obstacles. In the terminology of the theory of trapped modes, this would be deemed to be the lowest symmetric mode which may occur between the bodies. Numerical experimentation by colleagues at Bristol University shows that by putting the sources at other odd multiples of π/K apart, symmetric modes with more nodal lines are generated. However, this procedure generates new body contours and it has not yet been established whether, for a given pair of bodies, an infinite sequence of symmetric modes is possible. Antisymmetric modes between bodies have also be generated numerically by putting a wave source and a wave sink of equal strength an even multiple of π/K apart.

In the next section, a proof is given that the potential in (3.1) generates at least two streamlines which may be interpreted as two surface-piercing bodies.

4. Proof of the existence of a pair of surface-piercing bodies for which non-uniqueness can occur

The purpose of this section is to show that the potential given in (3.1) represents a solution to the boundary value problem stated in §2, for two surface-piercing bodies. The method used is to show that a streamline emanates from the free surface on the right of the source point at $x = a$, passes through the fluid below the source and re-enters the free surface on the left of the source point in the region $0 < x < a$ and thus removes the source point from the fluid. As the potential is symmetric about $x = 0$, the streamline pattern is also symmetric and so, once the first streamline is shown to exist, a second streamline must exist around the source point at $(-a, 0)$.

From (3.7), the value of the stream function on the free surface in the region $x > a$, where $Ka = \pi/2$, is given by

$$\psi(x,0) = \mathrm{Im}\left[\mathrm{i}e^{\mathrm{i}Kx}(\mathrm{E}_1(\mathrm{i}Kx + \mathrm{i}\pi/2) - \mathrm{E}_1(\mathrm{i}Kx - \mathrm{i}\pi/2))\right], \quad x > a. \tag{4.1}$$

When the argument of the exponential integral is purely imaginary, it is convenient to express it in terms of the Sine and Cosine integrals. A rearrangement of (5.2.21) and (5.2.23) in Abramowitz & Stegun (1965) gives

$$\mathrm{E}_1(\mathrm{i}X) = \mathrm{i}\left[\mathrm{Si}(X) - \pi/2\right] - \mathrm{Ci}(X), \quad X > 0 \tag{4.2}$$

and

$$\mathrm{E}_1(-\mathrm{i}X) = -\mathrm{i}\left[\mathrm{Si}(X) - \pi/2\right] - \mathrm{Ci}(X), \quad X > 0 \tag{4.3}$$

where $\mathrm{Si}(X)$ and $\mathrm{Ci}(X)$ are the Sine and Cosine integral respectively, defined by

$$\mathrm{Si}(X) = \int_0^X \frac{\sin t}{t}\,\mathrm{d}t \tag{4.4}$$

and

$$\mathrm{Ci}(X) = -\int_X^\infty \frac{\cos t}{t}\,\mathrm{d}t. \tag{4.5}$$

Both of the arguments of the exponential integrals in (4.1) are positive imaginary for $x > a$ and so substitution of (4.2) into (4.1) yields, after some manipulation,

$$\begin{aligned}\psi(x,0) &= [\mathrm{Si}(Kx - \pi/2) - \mathrm{Si}(Kx + \pi/2)]\sin Kx \\ &\quad + [\mathrm{Ci}(Kx - \pi/2) - \mathrm{Ci}(Kx + \pi/2)]\cos Kx, \quad x > a.\end{aligned} \tag{4.6}$$

In order to consider a streamline emanating from this part of the free surface it is first necessary to determine the range of values which ψ takes in this region. There is a logarithmic singularity in $\mathrm{Ci}(X)$ at $X = 0$ but as $\cos Kx = -(Kx - \frac{1}{2}\pi) + O((Kx - \frac{1}{2}\pi)^3)$ as $Kx \to \frac{1}{2}\pi$ and $\mathrm{Si}(0) = 0$,

$$\psi(x,0) \to -\mathrm{Si}(\pi) \approx -1.852 \qquad \text{as} \qquad x \to a^+. \tag{4.7}$$

Both $\mathrm{Si}(X)$ and $\mathrm{Ci}(X)$ are bounded as $X \to \infty$ and so

$$\psi(x,0) \to 0 \qquad \text{as} \qquad x \to \infty. \tag{4.8}$$

Thus $\psi(x,0)$ varies from $-\mathrm{Si}(\pi)$ at $x = a^+$ to 0 as $x \to \infty$. Differentiation of ψ using the representation in (4.6) yields, after some manipulation,

$$\frac{\partial\psi}{\partial x}(x,0) = K\left[g\left(Kx - \tfrac{1}{2}\pi\right) + g\left(Kx + \tfrac{1}{2}\pi\right)\right], \qquad x > a, \tag{4.9}$$

where $g(X)$ is the auxiliary function defined by Abramowitz & Stegun (1965, 5.2.7 and 5.2.13) as

$$g(X) = -\mathrm{Ci}(X)\cos X - \left(\mathrm{Si}(X) - \tfrac{1}{2}\pi\right)\sin X = \int_0^\infty \frac{t\mathrm{e}^{-Xt}}{1+t^2}\,\mathrm{d}t, \qquad X > 0. \tag{4.10}$$

Clearly, $g(X) > 0$ for $X > 0$ and so from (4.9), $\partial\psi/\partial x(x,0) > 0$ for $x > a$ and the stream function increases strictly monotonically from $-\mathrm{Si}(\pi)$ to 0 as x varies from a^+ to infinity.

The value of the stream function at the point $(x_0, 0)$, where $a < x_0 < \infty$, is denoted by $\psi = \psi_0$, where $-\mathrm{Si}(\pi) < \psi_0 < 0$. This is not an isolated point at which $\psi = \psi_0$ as can be demonstrated by considering the variation of ψ along a short line which joins the point $(x_0(1-\epsilon), 0)$ to the point $(x_0(1+\epsilon), 0)$ and passes through the interior of the fluid beneath the point $(x_0, 0)$, where $0 < \epsilon \ll 1$ and $x_0(1-\epsilon) > a$. As $\psi(x,0)$ is strictly monotonically increasing in $a < x < \infty$ and $\psi(x,y)$ is continuous in $y \geqslant 0$ (except at the source points) ψ must vary continuously along the line from a value $\psi < \psi_0$ at $(x_0(1-\epsilon), 0)$ to a value $\psi > \psi_0$ at $(x_0(1+\epsilon), 0)$. Thus, ψ must take the value ψ_0 on the line at some point in the interior of the fluid. As ϵ and the length of the line may be made arbitrarily small, this means that there are points in the interior of the fluid, arbitrarily close to $(x_0, 0)$, at which $\psi = \psi_0$ and so a streamline on which $\psi = \psi_0$ emanates from $(x_0, 0)$ into the fluid.

A similar argument may be used to show that the streamline cannot end in the fluid. The contrary is supposed, namely that the line terminates in the fluid. The mean value theorem for harmonic functions is applied to ψ on a circle of radius δ, where $0 < \delta \ll 1$, centred at the end of the line. The theorem implies that either $\psi = \psi_0$ everywhere on the circle or ψ takes values both greater than ψ_0 and less than ψ_0 and thus, by continuity, takes the value ψ_0 at least twice on the circle. In either case, there are at least two points on the circle at which $\psi = \psi_0$. One point is on the original streamline but, as δ can be made arbitrarily small, the other point forms a continuation of the streamline, which contradicts the supposition that the streamline terminates.

Thus, the streamline must either go off to infinity or re-enter the free surface (possibly at one of the source points). The asymptotic form of the exponential integral is given by Abramowitz & Stegun (1965, 5.1.51) as

$$\mathrm{E}_1(z) = \frac{\mathrm{e}^{-z}}{z}\left[1 + O\left(\frac{1}{z}\right)\right] \qquad \text{as} \quad |z| \to \infty, \qquad |\arg z| < \frac{3\pi}{2} \tag{4.11}$$

and so, from the definition of ψ in (3.7),

$$\psi(x,y) \to 0 \qquad \text{as} \qquad x^2 + y^2 \to \infty, \quad y \geqslant 0 \tag{4.12}$$

and so a streamline on which $\psi = \psi_0 < 0$ cannot go off to infinity. The streamline must, therefore, re-enter the free surface. It cannot re-enter the free surface at the point $(x_0, 0)$ at which it left because if it did, the streamline would be closed and an application of the maximum principle for harmonic functions would yield that $\psi = \psi_0$ everywhere in the region contained within that streamline. Analytic continuation of harmonic functions would then mean that $\psi = \psi_0$ everywhere in the fluid (except at the source points) which violates the definition of ψ. Furthermore, the streamline cannot re-enter the free surface at any other point in the range $a < x < \infty$ because $\psi(x,0)$ is strictly monotonically increasing in this region and so there is only one point, namely $(x_0,0)$, at which $\psi = \psi_0$.

The small-argument expansion of $\mathrm{E}_1(z)$ is given by Abramowitz & Stegun (1965, 5.1.11) as

$$\mathrm{E}_1(z) = -\gamma - \ln z + O(z) \qquad \text{as} \quad |z| \to 0, \qquad |\arg z| < \pi, \tag{4.13}$$

where γ is Euler's constant. Thus, from the definition of ψ in (3.7) and the relationship between the exponential integral and the Sine and Cosine integrals in (4.2),

$$\psi(x,y) = \alpha - \mathrm{Si}(\pi) + o(1) \qquad \text{as} \qquad (x-a)^2 + y^2 \to 0, \quad y \geqslant 0, \tag{4.14}$$

where $\alpha = \arg(Kx - \pi/2 - \mathrm{i}Ky)$ and α varies in the interior of the fluid from $-\pi$ on $y = 0$, $x < a$ to 0 on $y = 0$, $x > a$. Thus the values of the stream function on the streamlines which enter the source point at $(a,0)$ from the region $y \geqslant 0$ are in the range $-\mathrm{Si}(\pi) - \pi \leqslant \psi \leqslant -\mathrm{Si}(\pi)$ and so the streamline on which $\psi = \psi_0$ where $-\mathrm{Si}(\pi) < \psi_0 < 0$ cannot go into the source point.

From (3.6) the stream function is antisymmetric about the line $x = 0$ and so $\psi = 0$ on this line and the streamline on which $\psi = \psi_0 < 0$ cannot cross the line $x = 0$. The only remaining possibility is that the streamline on which $\psi = \psi_0$ re-enters the free surface in the region $0 < x < a$. Because the stream function is antisymmetric, $\psi = 0$ at the point $(0,0)$ and from the behaviour of stream function near the source point given in (4.14), $\psi(x,0) \to -\mathrm{Si}(\pi) - \pi$ as $x \to a^-$. As $\psi(x,0)$ is continuous in $0 < x < a$, it must take all values in the range $-\mathrm{Si}(\pi) - \pi < \psi < 0$ at least once. Thus, there is at least one point in this range at which $\psi = \psi_0$ and the streamline re-enters the free surface at such a point.

It has been demonstrated, therefore, that there exists at least one streamline which connects the free surface on either side of the source point at $(a, 0)$ and removes it from the fluid domain. By symmetry, the streamline on which $\psi = -\psi_0$ connects the free surface on either side of the source point at $(-a, 0)$. These streamlines may be interpreted as body boundaries and so the potential given in (3.1) represents a trapped mode of oscillation between two surface-piercing bodies. There is, in fact, a double infinity of possible pairs of bodies for which this potential represents a trapped mode and these bodies are constructed from the streamlines on which $\psi = \psi_0$ and $\psi = -\psi_1$, where $-\mathrm{Si}(\pi) < \psi_i < 0$, $i = 0, 1$.

5. Conclusion

In this work, an example of non-uniqueness in the two-dimensional water wave problem has been generated by constructing a potential which does not radiate waves to infinity and interpreting two of the streamlines of the flow as body boundaries. The potential is constructed from two wave sources which are placed in the free surface and separated by a distance of half a wavelength. It has been shown both analytically and numerically that this potential represents a trapped mode of oscillation between two surface-piercing bodies. Work is currently under way to try and determine whether such a mode exists for an arbitrary pair of surface-piercing bodies or whether geometric restrictions on the bodies are required. In addition, the question of whether a trapped mode can exist at more than one frequency for a given pair of bodies is under investigation. Preliminary numerical evidence indicates that it is possible to construct bodies for which trapped modes with more nodal lines exist, but it is not yet clear whether more than one mode can exist for a given pair of bodies. The corresponding three-dimensional problem is being considered by a colleague at Loughborough University and the early numerical indications are that it is possible to generate a trapped mode of oscillation within a surface-piercing, torus-like body, by constructing the potential from a suitably positioned ring source. An attempt to prove these numerical results is under way.

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